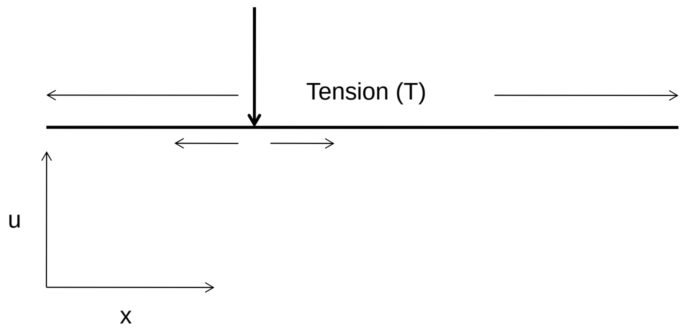
2 Basic Seismological Theory

- Waves on a string
- Stress and strain
- Seismic waves
- Snell's law
- Plane wave reflection and transmission coefficients
- Surface waves
- Dispersion
- Normal modes of the earth

Setting

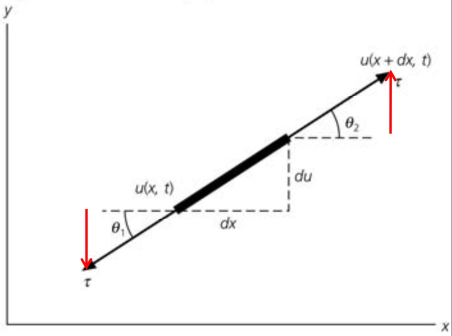


The displacement u(x,t) as a function of both position along the string and of time

F = m**a**

$$F(x, t) = \underline{\tau \sin\theta_2 - \tau \sin\theta_1} = \rho dx \frac{\partial^2 u(x, t)}{\partial t^2}$$

Figure 2.2-1: Tensions on a string segment.



For θ small, $\sin\theta \approx \theta \approx \tan\theta$:

$$\tau\left(\frac{\partial u(x + dx, t)}{\partial x} - \frac{\partial u(x, t)}{\partial x}\right) = \rho dx \frac{\partial^2 u(x, t)}{\partial t^2}$$

Discarding the higher Taylor series terms:

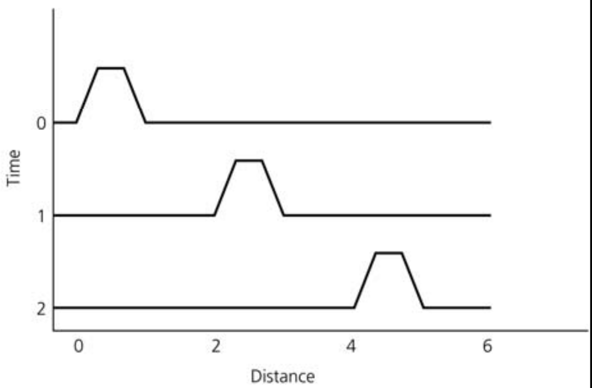
$$\tau\left(\frac{\partial u(x, t)}{\partial x} + \frac{\partial^2 u(x, t)}{\partial x^2} dx - \frac{\partial u(x, t)}{\partial x}\right) = \tau \frac{\partial^2 u(x, t)}{\partial x^2} dx = \rho dx \frac{\partial^2 u(x, t)}{\partial t^2}$$

Wave equation:

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{1}{\mathrm{v}^2} \frac{\partial^2 u(x, t)}{\partial t^2}$$

where $\mathrm{v} = (\tau/\rho)^{1/2}$

Figure 2.2-2: Propagating pulse, $f(x - 2t)$.



$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{1}{\mathrm{v}^2} \frac{\partial^2 u(x, t)}{\partial t^2}$$

The solution has the form $u(x, t) = f(x \pm \mathrm{v}t)$:

$$\frac{\partial^2 u(x, t)}{\partial x^2} = f''(x \pm \mathrm{v}t) \quad \text{and} \quad \frac{\partial^2 u(x, t)}{\partial t^2} = \mathrm{v}^2 f''(x \pm \mathrm{v}t)$$

where f'' is the second derivative of f

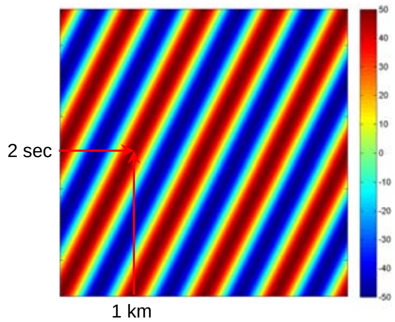
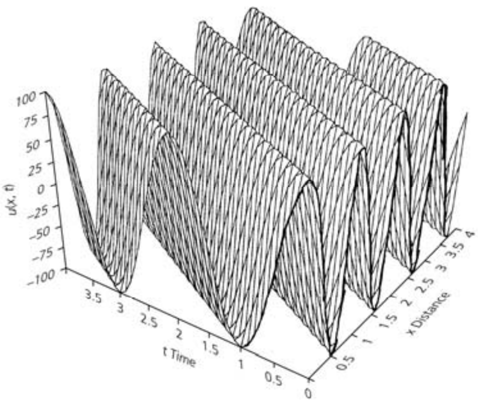


Figure 2.2-3: Displacement field of $u = A \cos(\pi t - 2\pi x)$.



Harmonic (sinusoidal) function:

$$u(x, t) = Ae^{i(\omega t \pm kx)} = A\cos(\omega t \pm kx) + Ai\sin(\omega t \pm kx)$$

$\omega t = kx \rightarrow$ v = x/t = ω/k

gives v = ω/k

Complex numbers

A complex number can be written in *polar coordinate* form as

$$z = a + ib = re^{i\theta} = r(\cos\theta + i\sin\theta).$$

$$r = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}(b/a).$$

$$a = r\cos\theta \qquad b = r\sin\theta.$$

The conjugate of a complex number z, z^*:

$$z^* = a - ib = r\cos\theta - ir\sin\theta = r\cos(-\theta) + ir\sin(-\theta) = re^{-i\theta},$$

$$|z|^2 = z\,z^* = (a + ib)(a - ib) = (a^2 + b^2)$$

$$= re^{i\theta}\, re^{-i\theta} = r^2.$$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = (e^{i\theta} + e^{-i\theta})/2 \quad \text{and} \quad \sin\theta = (e^{i\theta} - e^{-i\theta})/2i.$$

Figure A.2-1: Representation of a complex number.

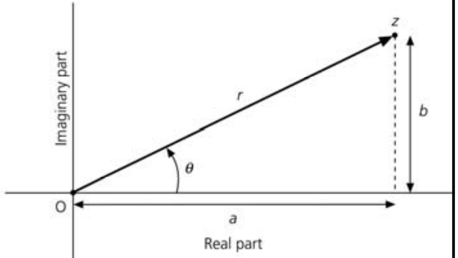


Table 2.2-1 Relationships Between Wave Variables

QUANTITY	UNITS	
Velocity	distance/time	$v = \omega/k = f\lambda = \lambda/T$
Period	time	$T = 2\pi/\omega = 1/f = \lambda/v$
Angular Frequency	time^{-1}	$\omega = 2\pi/T = 2\pi f = kv$
Frequency	time^{-1}	$f = \omega/(2\pi) = 1/T = v/\lambda$
Wavelength	distance	$\lambda = 2\pi/k = v/f = vT$
Wavenumber	distance^{-1}	$k = 2\pi/\lambda = \omega/v = 2\pi f/v$

Figure 2.2-4: Harmonic wave, $u = A\cos(\omega t - kx)$.

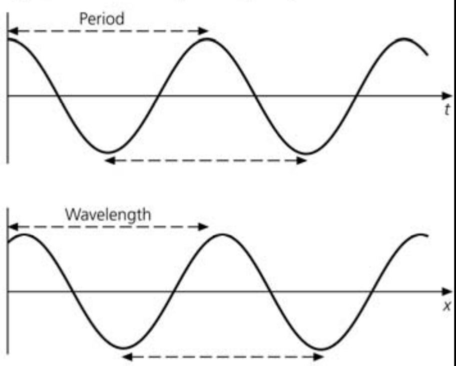
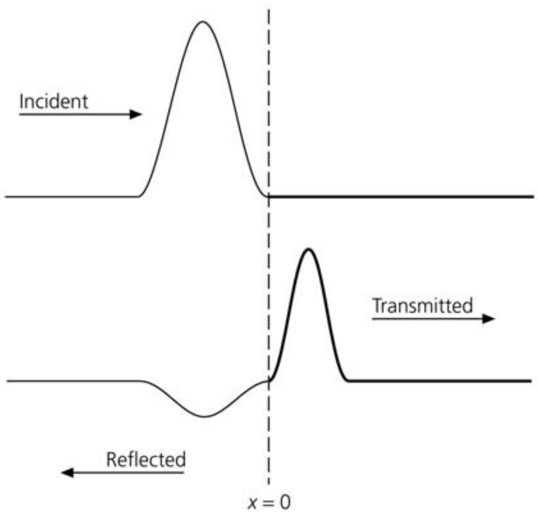


Figure 2.2-5: Transmitted and reflected wave pulses.



Left side:

$$u_1(x, t) = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

Positive x Negative x

Right side:

$$u_2(x, t) = Ce^{i(\omega t - k_2 x)}$$ Positive x only

Boundary conditions

- Continuity in displacement
  - $u_1(0,t) = u_2(0,t)$
- Continuity in tension
  - $\tau\sin\theta_1 = \tau\sin\theta_2$

Left side:

$$u_1(x, t) = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

Right side:

$$u_2(x, t) = Ce^{i(\omega t - k_2 x)}$$

$$u_1(0, t) = u_2(0, t)$$

$$Ae^{i\omega t} + Be^{i\omega t} = Ce^{i\omega t}$$

$$A + B = C$$

Left side:

$$u_1(x, t) = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

Right side:

$$u_2(x, t) = Ce^{i(\omega t - k_2 x)}$$

$$\tau \frac{\partial u_1(0, t)}{\partial x} = \tau \frac{\partial u_2(0, t)}{\partial x}$$

$$\tau k_1(A - B) = \tau k_2 C$$

Because the velocities on the two sides are $\mathrm{v}_i = (\tau/\rho_i)^{1/2}$ and $k_i = \omega/\mathrm{v}_i$,

$$\rho_1 \mathrm{v}_1 \, (A - B) = \rho_2 \mathrm{v}_2 C$$

$$A + B = C$$

$$\rho_1 \mathrm{v}_1 \, (A - B) = \rho_2 \mathrm{v}_2 C$$

Reflection coefficient:

$$R_{12} = \frac{B}{A} = \frac{\rho_1 \mathrm{v}_1 - \rho_2 \mathrm{v}_2}{\rho_1 \mathrm{v}_1 + \rho_2 \mathrm{v}_2}$$

ρv= ‘acoustic impedance’

Transmission coefficient:

$$T_{12} = \frac{C}{A} = \frac{2\, \rho_1 \mathrm{v}_1}{\rho_1 \mathrm{v}_1 + \rho_2 \mathrm{v}_2}$$

$$R_{12} = -R_{21} \qquad T_{12} + T_{21} = 2$$

$$R_{12} = \frac{B}{A} = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2}$$

Fixed end?

$$R_{fixed} = \frac{\rho_1 v_1 - \infty}{\rho_1 v_1 + \infty} = -1$$

Free end?

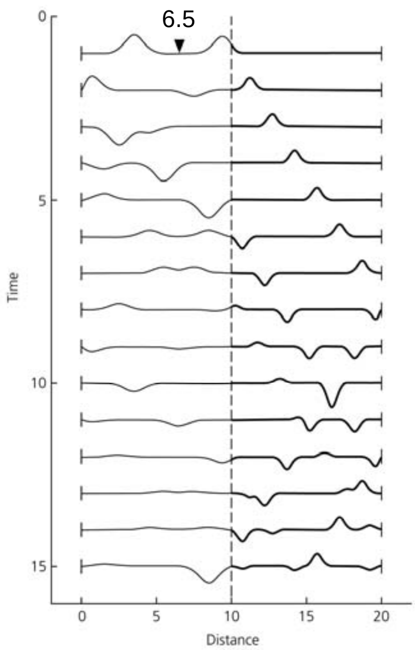
$$R_{free} = \frac{\rho_1 v_1 - 0}{\rho_1 v_1 + 0} = 1$$

Polarity?

Figure 2.2-6: Waves on a string of two different densities.

$\rho_1 = 1$, $v_1 = 3$
$\rho_1 v_1 = 3$

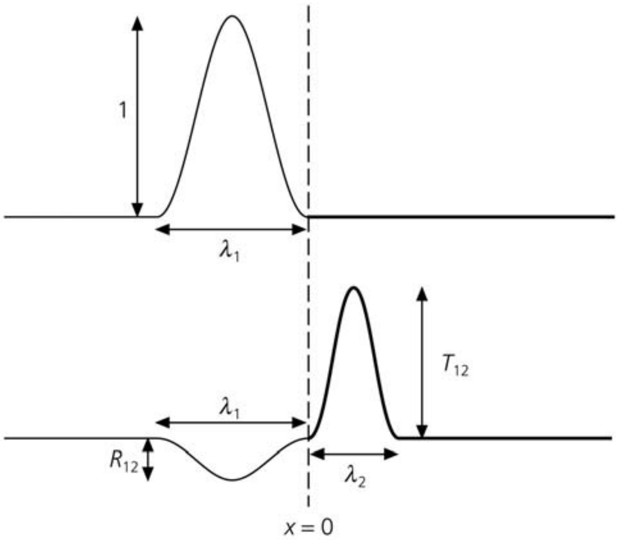
$R_{12} = -0.33$, $T_{12}=0.67$



$\rho_2 = 4$, $v_2 = 1.5$
$\rho_2 v_2 = 6$

$R_{21} = 0.33$, $T_{21}=1.33$

Figure 2.2-7: Reflected and transmitted amplitudes.



$$\omega = v_1 k_1 = v_2 k_2 = v_1 2\pi/\lambda_1 = v_2 2\pi/\lambda_2$$

2.4.4 Energy in a harmonic wave

T can be larger than 1, Is energy conserved?

Kinetic energy:

$$KE = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$$

because the mass of the spring is $m = \rho dx$

Averaged over one wavelength, with $u(x, t) = A\cos(\omega t - kx)$:

$$KE = \frac{\rho}{2\lambda}\int_0^\lambda \left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{\rho A^2 \omega^2}{2\lambda}\int_0^\lambda \sin^2(\omega t - kx)dx$$

Identity:

$$\int_0^\lambda \sin^2(\omega t - kx)dx = \lambda/2$$

$$KE = A^2\omega^2\rho/4$$

Potential energy:

strain: "The ratio of the change in the length to the original length"

$$e = \frac{(dx^2 + du^2)^{1/2} - dx}{dx} = \left[1 + \left(\frac{du}{dx}\right)^2\right]^{1/2} - 1 = \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2$$

(using the Taylor series approximation $(1 + a^2)^{1/2} \approx 1 + a^2/2$ for small a)

$$PE = \int_0^L e\tau\, dx = \frac{\tau}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$PE = \frac{\tau}{2\lambda}\int_0^\lambda \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{\tau\ A^2\ k^2}{2\lambda}\int_0^\lambda \sin^2(\omega t - kx)dx$$

$$PE = \tau A^2 k^2/4 = A^2\omega^2\rho/4 \qquad v^2 = \tau/\rho,\quad \tau = \frac{\omega^2\rho}{k^2}$$

$$KE = A^2\omega^2\rho/4$$

$$PE = A^2\omega^2\rho/4$$

Total energy:

$$E = PE + KE = A^2\omega^2\rho/2$$

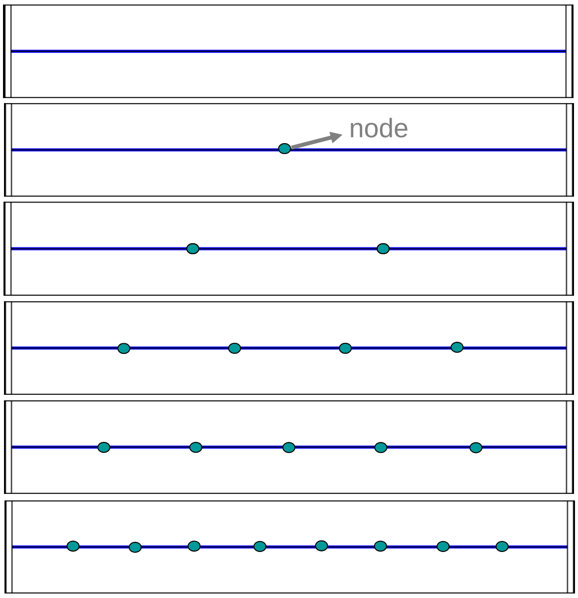
Energy flux:

$$\dot{\mathbf{E}} = A^2\omega^2\rho \mathrm{v}/2$$

$$\dot{\mathbf{E}}_R + \dot{\mathbf{E}}_T = R_{12}^2\omega^2\rho_1\mathrm{v}_1/2 + T_{12}^2\omega^2\rho_2\mathrm{v}_2/2$$

$$= (\omega^2/2)\ [R_{12}^2\mathrm{v}_1\rho_1 + T_{12}^2\mathrm{v}_2\rho_2] = \omega^2\rho_1\mathrm{v}_1/2 = \dot{\mathbf{E}}_I$$

Standing waves on a String



The string has a set of natural patterns of vibration called normal modes

2.2.5 Normal Modes of a string

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

Propagating wave solution:

$$u(x,t) = A\cos(\omega t \pm kx)$$

Mode solution:

$$u(x,t) = U(x,\omega)\cos(\omega t)$$

$$\frac{\partial^2 U(x,\omega)}{\partial x^2} = \frac{-\omega^2}{\mathrm{v}^2}U(x,\omega)$$

One solution of this equation is

$$U(x,\omega) = \sin(\omega x/\mathrm{v})$$

$U(x, \omega) = \sin(\omega x/\mathrm{v})$

For fixed ends at $x = 0$ and $x = L$:

$U(0, \omega) = U(L, \omega) = 0$

At $x = L$:

$U(L, \omega) = \sin(\omega L/\mathrm{v}) = 0$

which occurs for angular frequencies ω_n such that

$\omega_n L/\mathrm{v} = n\pi$ or $\omega_n = n\pi\mathrm{v}/L.$

(Eigenfrequencies)

Eigenfrequencies:

$\omega_n L/\mathrm{v} = n\pi$ or $\omega_n = n\pi\mathrm{v}/L$

$u(x, t) = U(x, \omega)\cos(\omega t)$

$u(x, t) = U_n(x, \omega_n)\cos(\omega_n t)$

where

$U_n(x, \omega_n) = \sin(\omega_n x/\mathrm{v}) = \sin(n\pi x/L)$

is known as the spatial *eigenfunction.*

Because $\omega = \mathrm{v}k = \mathrm{v}2\pi/\lambda$, the eigenfrequencies correspond to

$\omega_n = n\pi\mathrm{v}/L = 2\pi\mathrm{v}/\lambda$ or $L = n\lambda/2$ L -> infinite, $W_{n+1} - W_n$ -> 0

Each spatial eigenfunction has an integral number of half wavelengths along the string's length L, so the displacement at both ends is zero. The solutions are standing waves, known as the *normal modes* or *free oscillations.*

Travelling wave:

$$u(x, t) = \sum_{n=0}^{\infty} A_n U_n(x, \omega_n) \cos(\omega_n t)$$

The modes are orthogonal:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\delta_{mn}$$

Source Location

$$A_n = \sin(n\pi x_s/L)F(\omega_n)$$

Time history of the source

$$u(x, t) = \sum_{n=0}^{\infty} \sin(n\pi x_s/L)\ F(\omega_n)\ \sin(n\pi x/L)\ \cos(\omega_n t)$$

$$u(x, t) = \sum_{n=0}^{\infty} \sin(n\pi x_s/L)\ F(\omega_n)\ \sin(n\pi x/L)\ \cos(\omega_n t)$$

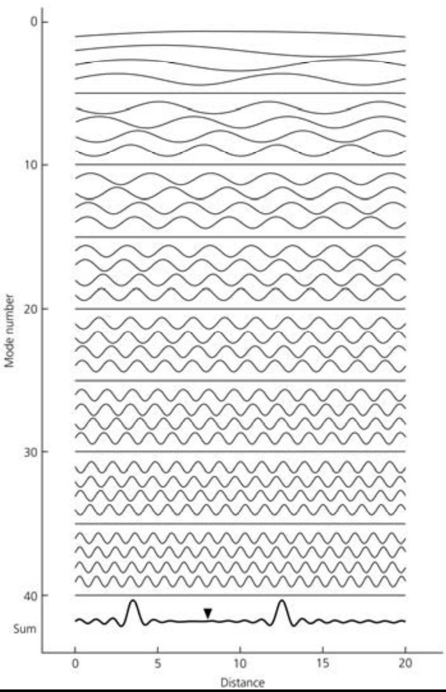
The source, at $x_s = 8$, is described by

$F(\omega_n) = \exp[-(\omega_n \tau)^2/4]$

with $\tau = 0.2$.

L=20, v=3, L0 = 8
Wn = $n\pi v/L$

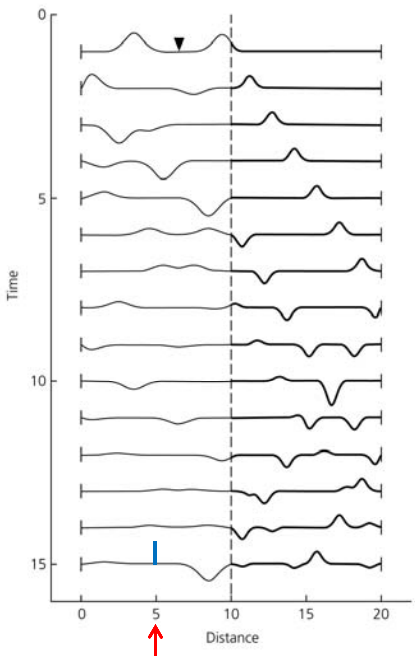
Figure 2.2-8: Waves on a string as a summation of modes.



$v_1 = 2$

$R_{12} = 0.5$, $T_{12}=1$

Figure 2.2-6: Waves on a string of two different densities.



$v_2 = 1$

$R_{21} = -0.5$, $T_{21}=1$

Review

- Solution of wave equation in 1D
  - Propagating wave solution
  - Normal mode (free oscillation solution)
    - Solution can be represented as weighted sum of modes (standing waves)
    - Eigenfrequency is depending on structure.
    - Excitation coefficient is depending on the source.
- Definitions of wavelength, period, …
- Reflection and transmission coefficient
- Energy conservation at boundary.

2.3 Stress and strain

Preparing for seismic waves in 3D:

Stress tensor
Strain tensor
Equations of motion
Constitutive equations

Continuum Mechanics:
Force --> per unit volume
Mass --> per unit volume

$\mathbf{F} = m\mathbf{a}$

As vector:

$$\mathbf{f}(\mathbf{x}, t) = \rho \frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2}$$

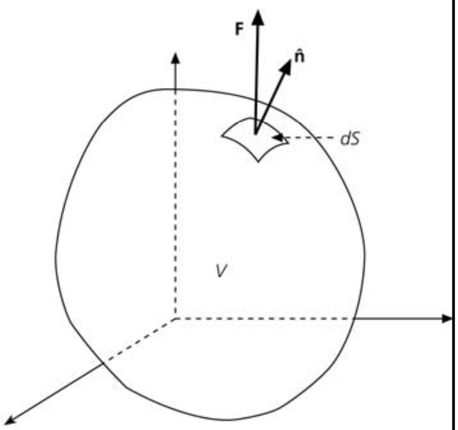
As components:

$$f_i(\mathbf{x}, t) = \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2} = \rho \ddot{u}_i$$

2.3.2 Stress

Stress is a measure of force per unit area in a body

Figure 2.3-1: Surface force on a volume element.



Two classes of forces:

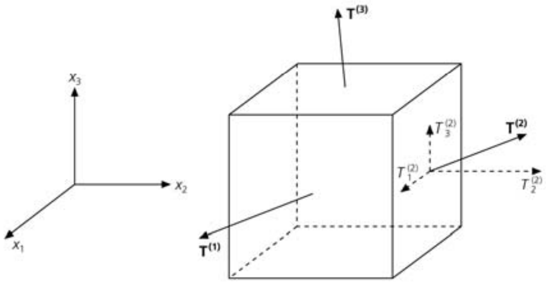
Body forces (e.g., gravity)

Surface forces (e.g., pressure underwater, stresses)

$$\mathbf{T}(\hat{\mathbf{n}}) = \lim_{dS \to 0} \frac{\mathbf{F}}{dS}$$

The traction vector has the same orientation as the force, and is a function of the unit normal vector $\hat{\mathbf{n}}$

Figure 2.3-2: Traction vectors on the faces of a volume element.



$\mathbf{T}^{(j)}$ is the traction vector acting on the surface whose outward normal is in the positive $\hat{\mathbf{e}}_j$ direction.

The components of the three traction vectors are $T_i^{(j)}$, where the upper index (j) indicates the surface and the lower (i) index indicates the component.

Example, $T_3^{(1)}$ is the x_3 component of the traction on the surface whose normal is $\hat{\mathbf{e}}_1$.

Stress tensor, σ_{ji}:

$$\sigma_{ji} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \end{pmatrix} = \begin{pmatrix} T_1^{(1)} & T_2^{(1)} & T_3^{(1)} \\ T_1^{(2)} & T_2^{(2)} & T_3^{(2)} \\ T_1^{(3)} & T_2^{(3)} & T_3^{(3)} \end{pmatrix}$$

Surface

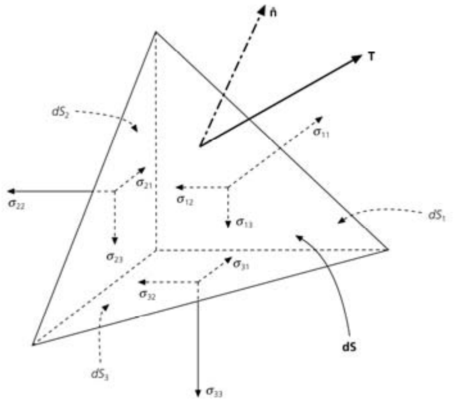
$$\sigma_{ij}$$

Component

The tensor's rows are the three traction vectors.

The stress is the force per unit area that the material on the outside of the surface (the side to which $\hat{\mathbf{n}}$ points) exerts on the material inside.

Figure 2.3-3: Stress components on the faces of a tetrahedron.

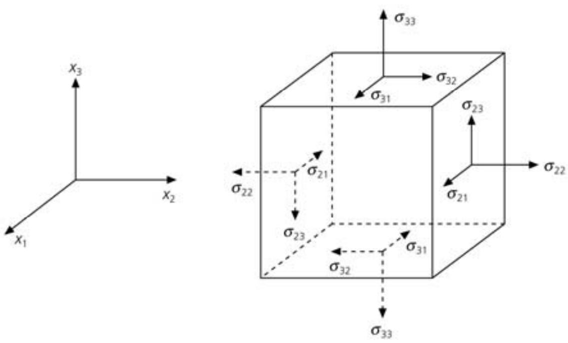


The stress tensor gives the traction vector **T** acting on any surface within the medium.

Example/ The traction on an arbitrary element of surface dS, whose normal $\hat{\mathbf{n}}$ is not along a coordinate axis, is found by multiplying each component of the traction by the area of the face it acts on and summing over the faces.

$$T_i = \sigma_{1i}n_1 + \sigma_{2i}n_2 + \sigma_{3i}n_3 = \sum_{j=1}^{3} \sigma_{ji}n_j = \sigma_{ji}n_j$$

Figure 2.3-4: Stress components on the faces of a volume element.



Normal stresses: σ_{11}, σ_{22}, σ_{33}

Shear stresses: σ_{12}, σ_{13}, σ_{21}, σ_{23}, σ_{31}, σ_{32}

Normal stresses are positive outward, and expand the volume.
---> *Tension*

Normal stresses are negative inward, and contract the volume.
---> *Compression*
(Holds true for Earth's interior)

The stress tensor is *symmetric:*

$\boxed{\sigma_{ij} = \sigma_{ji}}$

Otherwise: torques --> rotations!

Allows us to define tractions as

$$T_i = \sum_{j=1}^{3} \sigma_{ij} n_j = \sigma_{ij} n_j$$

(as components)

$$\mathbf{T} = \sigma \hat{\mathbf{n}}$$

(as vectors)

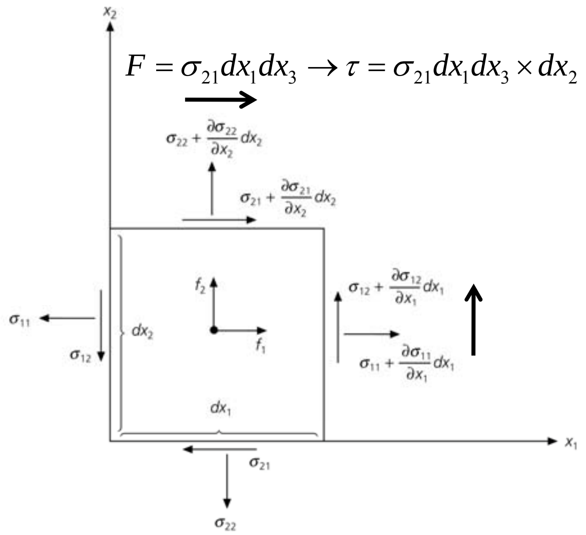
Stress = force/area

In cgs: 1 bar = 10^6 dyn/cm^2

1 atm = 1.01 bars

In SI: 1 Pascal (Pa) = 1 N/m^2 = 10^{-5} bars

Figure 2.3-5: Torques on a rectangle.



$$F = \sigma_{21} dx_1 dx_3 \rightarrow \tau = \sigma_{21} dx_1 dx_3 \times dx_2$$

$$F = \sigma_{12} dx_2 dx_3 \rightarrow \tau = \sigma_{12} dx_2 dx_3 \times dx_1$$

2.3.3 Stress as a tensor

What makes a vector a *vector*?

- It remains the same in different coordinate systems.
- Its components in different coordinate systems are related by a transformation matrix A.

$$\mathbf{u}' = A\mathbf{u}$$

Similarly, σ is a *tensor* (and not just a matrix) because it transforms between coordinates according to:

$$\sigma' = A \sigma A^T$$

2.3.3 Stress as a tensor

$$\mathbf{T}' = A\mathbf{T}, \quad \hat{\mathbf{n}}' = A\hat{\mathbf{n}}$$

$$\hat{\mathbf{n}} = A^{-1}\hat{\mathbf{n}}' = A^T\hat{\mathbf{n}}'$$

$$\mathbf{T}' = \sigma'\hat{\mathbf{n}}'$$

$$\mathbf{T}' = A\mathbf{T} = A\sigma\hat{\mathbf{n}} = A\sigma A^T\hat{\mathbf{n}}'$$

$$\sigma' = A\sigma A^T$$

Example: A block of material with faces perpendicular to the x_1 and x_2 axes is subject only to normal stresses σ_1 and σ_2, so the stress tensor is diagonal:

$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now consider a different block in the SAME PHYSICAL SITUATION, but with rotated sides:

$$\sigma' = A\sigma A^T$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma_1\cos^2\theta + \sigma_2\sin^2\theta & (\sigma_2 - \sigma_1)\sin\theta\cos\theta & 0 \\ (\sigma_2 - \sigma_1)\sin\theta\cos\theta & \sigma_1\sin^2\theta + \sigma_2\cos^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For example, if $\sigma_1 = 1$, $\sigma_2 = -1$, and $\theta = 45°$,

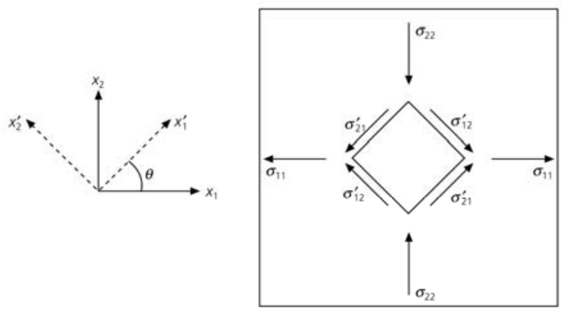
$$\sigma' = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

First block: only normal stresses.

Second block: only shear stresses.

Same state of stress!!
(but different coordinate axes!)

Figure 2.3-6: Different stress components in different coordinate systems.



2.3.4 Principal stresses

For any state of stress, a set of coordinate axes can be found that provides only normal stresses (and no shear stresses!).

These axes are called the *principal stress axes* and the normal stresses on these surfaces are called *principal stresses.*

To find the principal stresses, we use the concepts of eigenvalues and eigenvectors.

The shear components of the traction will be zero if the traction and normal vectors are parallel, such that they differ only by a multiplicative constant, λ,

$$T_i = \sigma_{ij} n_j = \lambda n_i$$

The principal stress axes $\hat{\mathbf{n}}$ are the eigenvectors of the stress tensor.

The principal stresses λ associated with each one are the eigenvalues.

Appendix A1

Eigenvalues and eigenvectors:

The product of an arbitrary $n \times n$ matrix A and an arbitrary n component vector $\mathbf{x}$

$$\mathbf{y} = A\mathbf{x}$$

is also a vector in n dimensions. This is not the same as coordinate transformation; the vector $\mathbf{x}$ is transformed into another distinct vector, with both vectors expressed in the same coordinate system.

A physically important class of such transformations are ones in which a vector is converted into one parallel to the original vector, so that

$$A\mathbf{x} = \lambda\mathbf{x}$$

where A is a matrix and λ is a scalar. The only effect of the transformation is that the length of $\mathbf{x}$ changes by a factor of λ.
For a given A, it is useful to know which vectors $\mathbf{x}$ and scalars λ satisfy this equation.

Appendix A2

$$(A - \lambda I)\,\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is a homogeneous system of linear equations, so nontrivial solutions exist only if the matrix $(A - \lambda I)$ is singular.
Seek values of λ such that the determinant

$$|\,(A - \lambda I)\,| = \det\begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{pmatrix} = 0$$

Evaluating the determinant gives the *characteristic polynomial*

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0,$$

which depends on three constants called the *invariants* of A:

$$I_1 = a_{11} + a_{22} + a_{33}$$

$$I_2 = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} + \det\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

$$I_3 = \det A.$$

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

The roots λ are the eigenvalues or principal stresses, σ_m.

In geology, where all stresses are compressive (negative), $|\,\sigma_1\,| \geq |\,\sigma_2\,| \geq |\,\sigma_3\,|$.

The eigenvalues gives the associated eigenvectors $\hat{\mathbf{n}}^{(m)}$, which are the three mutually perpendicular surfaces on which there are no tangential tractions.

$$A = \begin{pmatrix} \hat{\mathbf{n}}^{(1)} \\ \hat{\mathbf{n}}^{(2)} \\ \hat{\mathbf{n}}^{(3)} \end{pmatrix} = \begin{pmatrix} n_1^{(1)} & n_2^{(1)} & n_3^{(1)} \\ n_1^{(2)} & n_2^{(2)} & n_3^{(2)} \\ n_1^{(3)} & n_2^{(3)} & n_3^{(3)} \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

$$\sigma' = A\sigma A^T = \Lambda$$

$$\sigma = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\lambda = ?$$

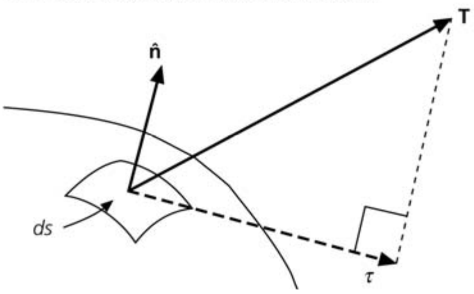
$$A = ?$$

$$\sigma' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ??$$

2.3.5 Maximum shear stress and faulting

Faulting will occur on the plane on which shear stress is highest (?)

Figure 2.3-7: Traction vector acting on a surface element.



The plane of maximum shear stress is located 45° from the maximum and minimum normal stress axes, and the value of the maximum shear stress is

$\tau = (\sigma_1 - \sigma_3)/2$

If the maximum and minimum principle stress axes are (1, 0, 0) and (0, 0, 1), the planes of maximum shear stress are

$\hat{\mathbf{n}} = \left(1/\sqrt{2},\ 0,\ 1/\sqrt{2}\right)$ and $\hat{\mathbf{n}} = \left(-1/\sqrt{2},\ 0,\ 1/\sqrt{2}\right)$

Because of the cohesive strength of rocks, fracture often occurs at about 25° from the maximum principle stress direction instead of 45°.

Figure 2.3-8: Compression of a rock sample.

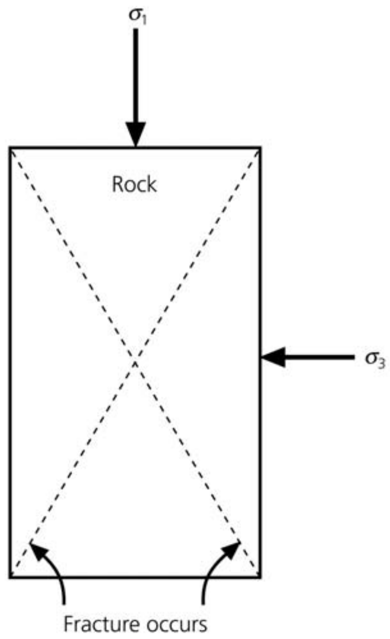
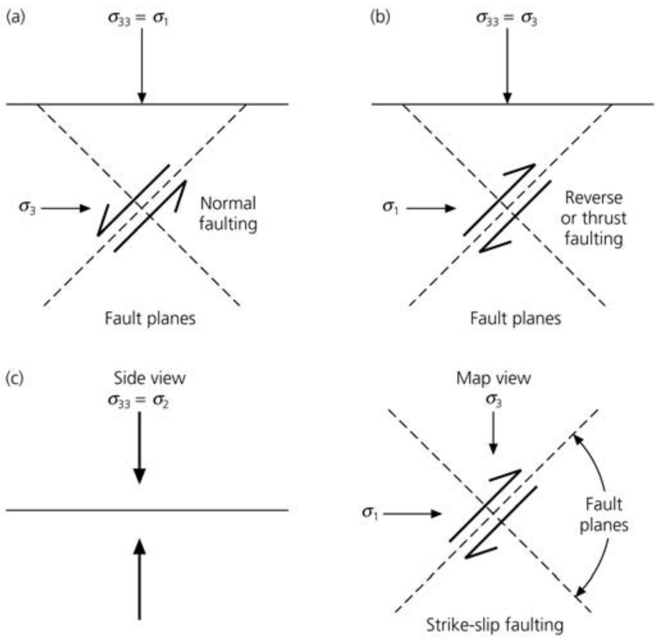


Figure 2.3-9: Stress fields associated with three types of faulting.



2.3.6 Deviatoric Stresses

Mean stress:

$$M = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3 = \sigma_{ii}/3$$

$$M = (\sigma_1 + \sigma_2 + \sigma_3)/3$$

Deviatoric stress tensor:

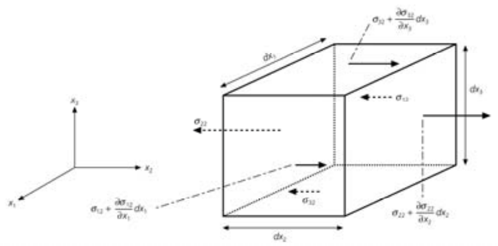
$$D_{ij} = \sigma_{ij} - M\delta_{ij}$$

$$D = \begin{pmatrix} \sigma_{11} - M & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - M & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - M \end{pmatrix}$$

For lithostatic conditions, $D = 0$.

2.3.7 Equation of motion

Figure 2.3-10: Stress components contributing to force in the x_2 direction.



Write $\mathbf{F} = m\mathbf{a}$ in terms of body and surface forces for each component.

For example, for the x_2 direction, the terms involving the $\hat{\mathbf{e}}_2$ and $-\hat{\mathbf{e}}_2$ faces (where the area of the faces are $dx_1 dx_3$,) are

$$\left[\sigma_{22}(\mathbf{x} + dx_2\hat{\mathbf{e}}_2) - \sigma_{22}(\mathbf{x})\right] dx_1 dx_3 = \left[\sigma_{22}(\mathbf{x}) + \frac{\partial \sigma_{22}(\mathbf{x})}{\partial x_2} dx_2 - \sigma_{22}(\mathbf{x})\right] dx_1 dx_3 = \frac{\partial \sigma_{22}(\mathbf{x})}{\partial x_2} dx_1 dx_2 dx_3$$

(Similar for the force in the x_2 direction due to the pairs of faces with normals $\pm\hat{\mathbf{e}}_1$ and $\pm\hat{\mathbf{e}}_3$.)

Summing the three terms, adding the body force component, and equating this net force to the density times this component of the acceleration yields

$$\left[\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3}\right] dx_1 dx_2 dx_3 + f_2\, dx_1 dx_2 dx_3 = \rho \frac{\partial^2 u_2}{\partial t^2}\, dx_1 dx_2 dx_3$$

$$\left[\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{32}}{\partial x_3}\right]dx_1dx_2dx_3+f_2\ dx_1dx_2dx_3=\rho\,\frac{\partial^2u_2}{\partial t^2}\ dx_1dx_2dx_3$$

$$\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{32}}{\partial x_3}+f_2=\sum_{j=1}^{3}\frac{\partial\sigma_{j2}}{\partial x_j}+f_2=\rho\,\frac{\partial^2u_2}{\partial t^2}$$

Similar equations apply for the x_1 and x_3 components of the force and acceleration:

$$\frac{\partial\sigma_{ji}(\mathbf{x},t)}{\partial x_j}+f_i(\mathbf{x},t)=\rho\,\frac{\partial^2u_i(\mathbf{x},t)}{\partial t^2}$$

$$\frac{\partial\sigma_{ij}(\mathbf{x},t)}{\partial x_j}+f_i(\mathbf{x},t)=\rho\,\frac{\partial^2u_i(\mathbf{x},t)}{\partial t^2}$$

$$\sigma_{ij,j}(\mathbf{x},t)+f_i(\mathbf{x},t)=\rho\ddot{u}_i(\mathbf{x},t)$$

This is the *Equation of motion*, which applies everywhere in a continuous medium.

$$\frac{\partial\sigma_{ij}(\mathbf{x},t)}{\partial x_j}+f_i(\mathbf{x},t)=\rho\,\frac{\partial^2u_i(\mathbf{x},t)}{\partial t^2}$$

Equation of equilibrium:
(accelerations are zero, like a static problem such as stresses resulting only from gravity)

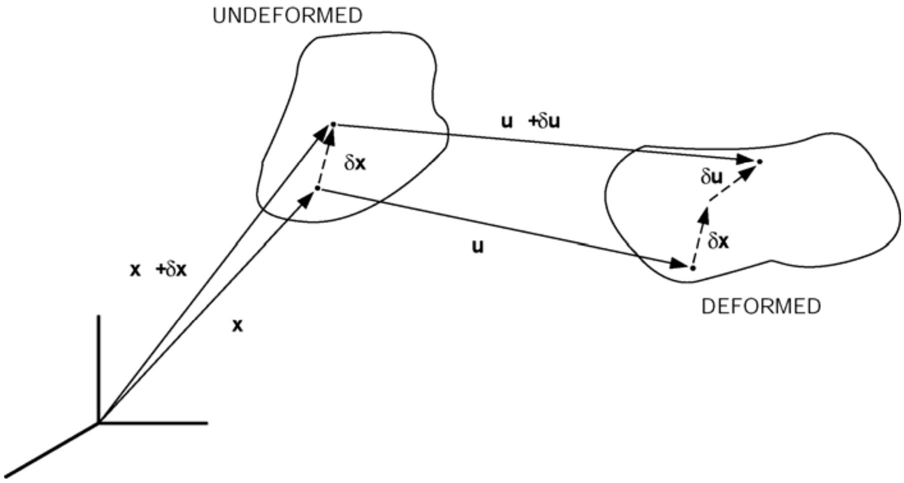
$$\sigma_{ij,j}(\mathbf{x},t)=-f_i(\mathbf{x},t)$$

Homogeneous equation of motion:
(with no forces, such as the harmonic oscillation of wave propagation)

$$\sigma_{ij,j}(\mathbf{x},t)=\rho\,\frac{\partial^2u_i(\mathbf{x},t)}{\partial t^2}$$

2.3.8 Strain

Figure 2.3-11: Change in relative displacement during deformation.



The *strain tensor* describes the deformation resulting from the differential motion within a body.

$$u_i(\mathbf{x} + \delta\mathbf{x}) \approx u_i(\mathbf{x}) + \frac{\partial u_i(\mathbf{x})}{\partial x_j}\,\delta x_j = u_i(\mathbf{x}) + \delta u_i$$

$$\delta u_i = \frac{\partial u_i(\mathbf{x})}{\partial x_j}\,\delta x_j$$

Although we are interested in deformation that distorts the body, there can also be a rigid body translation or a rigid body rotation, neither of which produces deformation. To distinguish these effects, we add and subtract $\partial u_j/\partial x_i$ and then separate it into two parts

$$\delta u_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\delta x_j = (e_{ij} + \omega_{ij})\delta x_j$$

ω_{ij} corresponds to a rigid body rotation without deformation. It is antisymmetric ($\omega_{ij} = -\omega_{ji}$), so the diagonal terms are zero.

Strain tensor:

$$e_{ij} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

Figure 2.3-12: Some possible strains for a two-dimensional element.

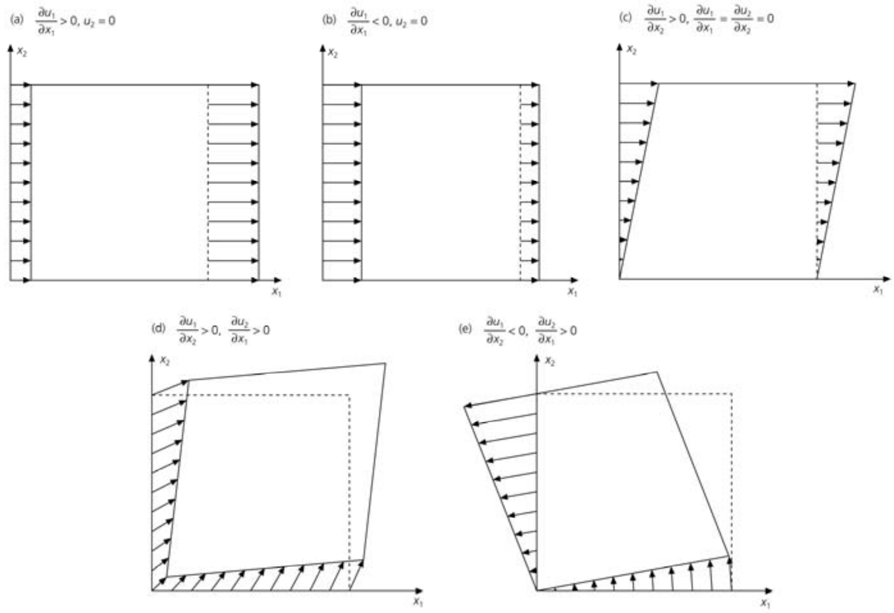
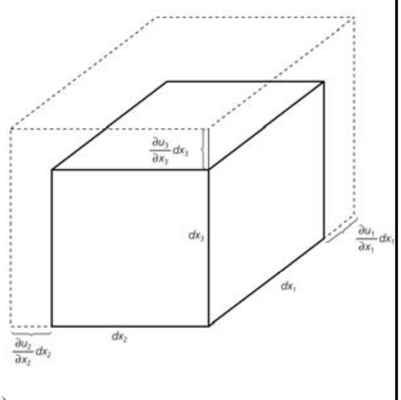


Figure 2.3-13: Change in volume due to principle strains.



The trace or sum of diagonal terms of the strain tensor is the *Dilatation*:
(gives the change in volume per unit volume associated with deformation)

$$\theta = e_{ii} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \mathbf{u}$$

For initial volume $dx_1 dx_2 dx_3$ the volume after deformation is

$$\left(1 + \frac{\partial u_1}{\partial x_1}\right)dx_1\left(1 + \frac{\partial u_2}{\partial x_2}\right)dx_2\left(1 + \frac{\partial u_3}{\partial x_3}\right)dx_3 \approx \left(1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)dx_1 dx_2 dx_3 = (1+\theta)dx_1 dx_2 dx_3$$

If the initial volume is $V = dx_1 dx_2 dx_3$, the final volume is $V + \Delta V = (1+\theta)V$,
so $\theta = \Delta V/V$

2.3.9 Constitutive equations

Constitutive equations give the relation between stress and strain.

The simplest type of materials are *linearly elastic,* such that there is a linear relation between the stress and strain tensors.

Others could describe viscous (Newtonian and non-Newtonian), viscoelastic, elastic-plastic, etc.

Linearly elastic constitutive equations gives rise to seismic waves.

Infinitesimal strain theory assumes very small displacements.

Example: a typical body wave can have displacements of 10 microns and wavelengths of 10 km. The resulting strain is about $(10^{-5}\text{m}/10^{4}\text{m}) = 10^{-9}$.

However, for strains greater than about 10^{-4}, the linear relation between stress and strain fails.

Linearly elastic material: The constitutive equation is *Hooke's law*:

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

The constants c_{ijkl}, the *elastic moduli,* describe the properties of the material.

Because the subscripts each range from 1 to 3, c_{ijkl} has 3^4, or 81 components.

The stress and strain tensors are symmetric

$$c_{ijkl} = c_{jikl} \qquad c_{ijkl} = c_{ijlk}$$

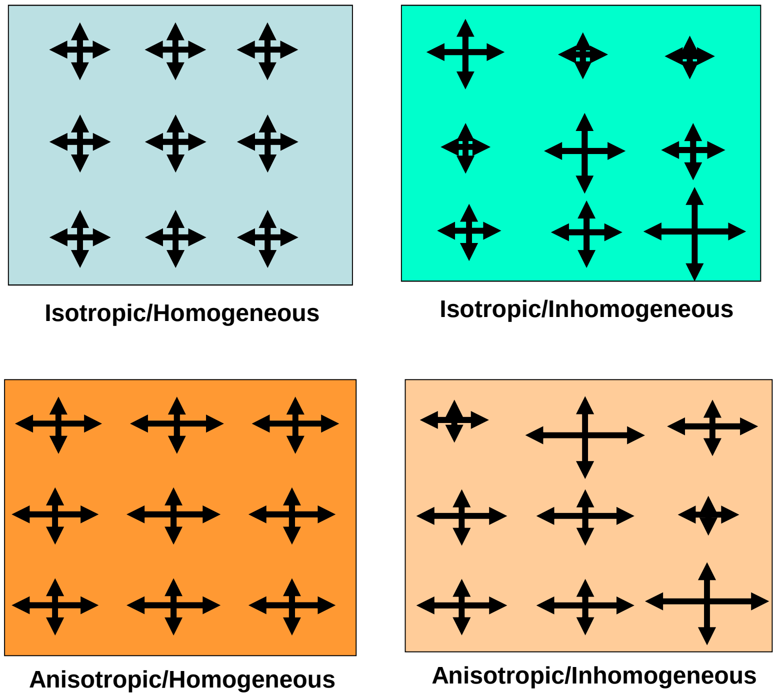
(brings the number of independent components to 36)

A further symmetry relation based on the idea of strain energy gives:

$$c_{ijkl} = c_{klij}$$

(...down to 21)

21 independent components are needed to describe general anisotropy.



Isotropy: Material behaves the same way regardless of orientation.

This reduces the number of independent c_{ijkl} to 2!!!!

One useful pair are the *Lame′ constants* λ and μ:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

Example:
$\sigma_{11} = \lambda\theta + 2\mu e_{11}$ and $\sigma_{12} = 2\mu e_{12}$.

Problem: λ has no physical meaning

More useful: μ and K

The *incompressibility* or *bulk modulus,* K is defined by subjecting a body to a lithostatic pressure dp, such that

$d\sigma_{ij} = -\,dp\delta_{ij}$

The resulting strains are $-dp\delta_{ij} = \lambda d\theta\delta_{ij} + 2\mu de_{ij}$

Set $i = j$ and sum ($\delta_{ii} = 3$): $-3dp = 3\lambda d\theta + 2\mu d\theta$ because $\delta_{ii} = 3$.

K is defined as the ratio of the pressure applied to the fractional volume change that results:

$$K = \frac{-dp}{d\theta} = \lambda + \frac{2}{3}\mu$$

The constitutive equation in terms of K and μ:

$\sigma_{ij} = K\theta\delta_{ij} + 2\mu\,(e_{ij} - \theta\delta_{ij}/3)$

Two parts: a volume change and a change in shape.

Two other elastic constants are defined by pulling the material along only one axis, leading to a state of stress called *uniaxial tension.* If the tension is applied along the x_1 axis:

$\sigma_{11} = (\lambda + 2\mu)e_{11} + \lambda e_{22} + \lambda e_{33}$

$\sigma_{22} = 0 = \lambda e_{11} + (\lambda + 2\mu)e_{22} + \lambda e_{33}$

$\sigma_{33} = 0 = \lambda e_{11} + \lambda e_{22} + (\lambda + 2\mu)e_{33}$

Subtracting the last two equations shows that $e_{22} = e_{33}$, so

$$e_{22} = e_{33} = \frac{-\lambda}{2(\lambda + \mu)}\,e_{11} = -\,\nu e_{11}$$

This defines Poisson's ratio, ν, which gives the ratio of the contraction along the other two axes to the extension along the axis where tension was applied.

Substituting this into the equation for σ_{11}:

$$\frac{\sigma_{11}}{e_{11}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} = E$$

E is called *Young's modulus,* the ratio of the tensional stress to the resulting extensional strain.

Relations between moduli

$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{(3K-\lambda)} = \frac{E}{2\mu} - 1 = \frac{3K-2\mu}{2(3K+\mu)} = \frac{3K-E}{6K}$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9K(K-\lambda)}{3K-\lambda} = 2\mu(1+\nu) = \frac{9K\mu}{3K+\mu} = 3K(1-2\nu)$$

$$K = \lambda + \frac{2}{3}\mu = \frac{\lambda(1+\nu)}{3\nu} = \frac{2\mu(1+\nu)}{3(1-2\nu)} = \frac{\mu E}{3(3\mu-E)} = \frac{E}{3(1-2\nu)}$$

$$\lambda = \frac{2\mu\nu}{1-2\nu} = \frac{\mu(E-2\mu)}{3\mu-E} = K - \frac{2}{3}\mu = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{3K\nu}{1+\nu} = \frac{3K(3K-E)}{9K-E}$$

$$\mu = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(K-\lambda) = \frac{E}{2(1+\nu)} = \frac{3K(1-2\nu)}{2(1+\nu)} = \frac{3KE}{9K-E}$$

Boundary Conditions:

At the free surface, stresses go to zero:

$T_i = \sigma_{ij} n_j = 0$

If the surface is horizontal,

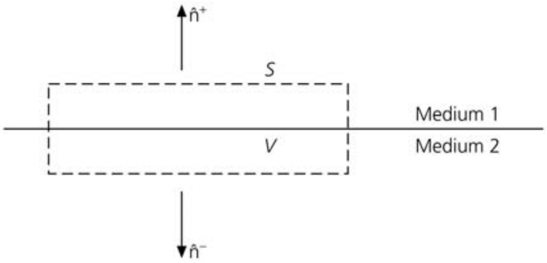
$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$

The components of the stress tensor that do not affect the tractions, in this case σ_{11}, σ_{12}, and σ_{22}, are unconstrained.

Similarly, no restriction is placed on the displacements.

A free surface corresponds in the one-dimensional case to a string whose end is free to move.

Figure 2.3-14: Gaussian pill box around a boundary.



Interface	Boundary Conditions
solid-solid	$T_i^+ = T_i^-$ $u_i^+ = u_i^-$
solid-liquid	$T_3^+ = T_3^-$ $T_2 = T_1 = 0$ $u_3^+ = u_3^-$
free surface	$T_i = 0$

Strain energy:

Energy in a compressed spring:

$$W = \int_0^x kx dx = \frac{1}{2} kx^2$$

By analogy, the strain energy stored in a volume is the integral of the product of stress and strain components summed: $W = \frac{1}{2} \int \sigma_{ij} e_{ij} dV = \frac{1}{2} \int c_{ijkl} e_{ij} e_{kl} dV$

The strain energy is symmetric in ij and kl, providing the symmetry that $c_{ijkl} = c_{klij}$.